5.2 Local Confluence and Critical Pairs

Dienstag, 8. Dezember 2015 09:00

Goal: Check whether a TRS is confluent.

Drawback: Confluence of TRSs is undecidable.

But: For terminating TRSs, confluence is decidable!

Reason: For terminating TRSs, it suffices to check the weaker property of <u>local confluence</u>.

<u>Def 5.2.1</u> (Local Confluence)

A relation $\to$ on a set M is <u>locally confluent</u> iff the following holds for all $s, t, p \in M$:

If $p \to s$ and $p \to t$,
then there exists a $q \in M$ such that $s \to^* q$ and $t \to^* q$.

If the black arrows hold,

$$\begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ s \arrow[dr, "*"'] & & t \arrow[dl, "*"] \\ & q & \end{tikzcd}$$

then the green arrows hold as well.

$$\text{Local Confluence} \;\underset{\not\Longrightarrow}{\overset{\text{trivial}}{\Longleftarrow}}\; \text{Confluence}$$

<u>Ex. 5.2.2</u> $\mathcal{R} = \{ b \to a, b \to c, c \to b, c \to d \}$.

$$a \leftarrow b \rightleftarrows c \to d$$

$\mathcal{R}$ is locally confluent: $b \to a$, $b \to c$. c and a are joinable, since $c \to^* a$.

$c \to b, c \to d$. b and d are joinable, since $b \to^* d$

$\mathcal{R}$ is not confluent: $b \to^* a$, $b \to^* d$, but b and d are not joinable

$c \to^* a$, $c \to^* d$, a and d not joinable

But: $\mathcal{R}$ is not terminating:
$b \to c \to b \to c \to \dots$

This problem can only occur for non-terminating TRSs. For terminating TRSs, local and full confluence is the same.

Thm 5.2.3 (Diamond Lemma, Newman's Lemma, Newman 1942)

Let $\to$ be a well-founded relation. Then $\to$ is locally confluent iff $\to$ is confluent.

Proof "$\Leftarrow$": Confluence trivially implies local confluence.

"$\Rightarrow$": Let $\to$ be locally confluent.
For all $p \in M$, we have to show:

means: s and t are joinable

If $s \,{}^*\!\!\leftarrow p \to^* t$, then $s \downarrow t$. $(*)$

Proof by Noetherian induction on p, using $\to$ as induction relation (possible, since $\to$ is well founded).
When proving $(*)$ for p, we can assume as ind. hypothesis

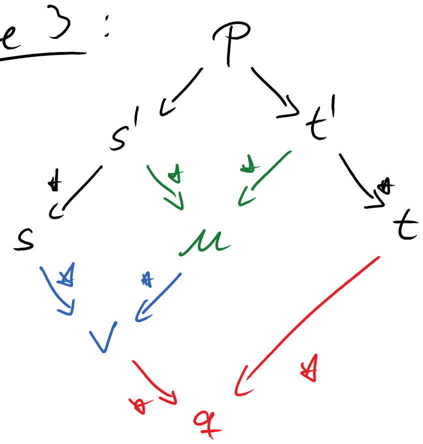
that it already holds for p' with $p \to p'$.

Let $s \xleftarrow{*} p \xrightarrow{*} t$.

<u>Case 1</u>: $s = p$. Then $s \downarrow t$, since $s = p \xrightarrow{*} t$.

<u>Case 2</u>: $p = t$. Then $s \downarrow t$, since $s \xleftarrow{*} p = t$

<u>Case 3</u>:



$s' \downarrow t'$ by local confluence

$s \downarrow u$ by the ind. hypothesis, because $p \to s'$

$v \downarrow t$ by the ind. hypothesis, because $p \to t'$

$\boxed{\text{q.e.d.}}$

For terminating TRSs, it is enough to check local confluence.

Now we show that for terminating TRSs, local confluence is decidable.

We have to investigate all critical situations:

$$\begin{tikzcd} & p \arrow[dl, "l_1 \to r_1"'] \arrow[dr, "l_2 \to r_2"] & \\ s & & t \end{tikzcd}$$

There can be infinitely many such situations. We want to reduce the check to just finitely many such situations.

$\Rightarrow$ Analyze possible critical situations in more detail.

There are rules $l_1 \to r_1$, $l_2 \to r_2$, substitutions σ_1, σ_2, and positions π_1, π_2 such that

- $p|_{\pi} = l_1\sigma_1$ and $p[r_1\sigma_1]_{\pi_1}$

- $p|_{\pi_1} = l_1\sigma_1$ and $\underbrace{p[r_1\sigma_1]_{\pi_1}}_{s}$
- $p|_{\pi_2} = l_2\sigma_2$ and $\underbrace{p[r_2\sigma_2]_{\pi_2}}_{t}$

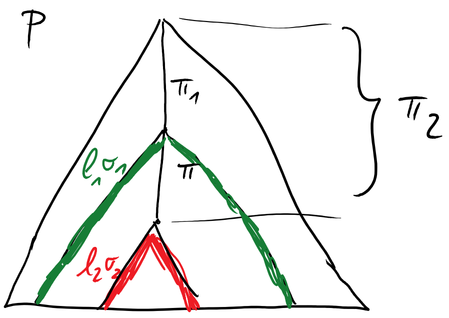
We have to consider the relationship between π_1 and π_2.

Case 1: π_1 and π_2 are independent ($\pi_1 \perp \pi_2$)

$$\begin{tikzcd}[column sep=tiny]
& \overbrace{f(\underline{f(x,i(x))},\underline{f(x,e)})}^{p} \arrow[dl] \arrow[dr] & \\
\underbrace{f(e,f(x,e))}_{s} \arrow[dr] & & \underbrace{f(f(x,i(x)),x)}_{t} \arrow[dl] \\
& f(e,x) &
\end{tikzcd}$$

This indeterminism is "harmless", i.e., it can always be joined.

Case 2: π_1 is above π_2 ($\pi_2 \geq_{\mathbb{N}^*} \pi_1$, i.e., there is a π such that $\pi_2 = \pi_1\pi$)



$p|_{\pi_1} = l_1\sigma_1$

$p|_{\pi_2} = l_2\sigma_2$

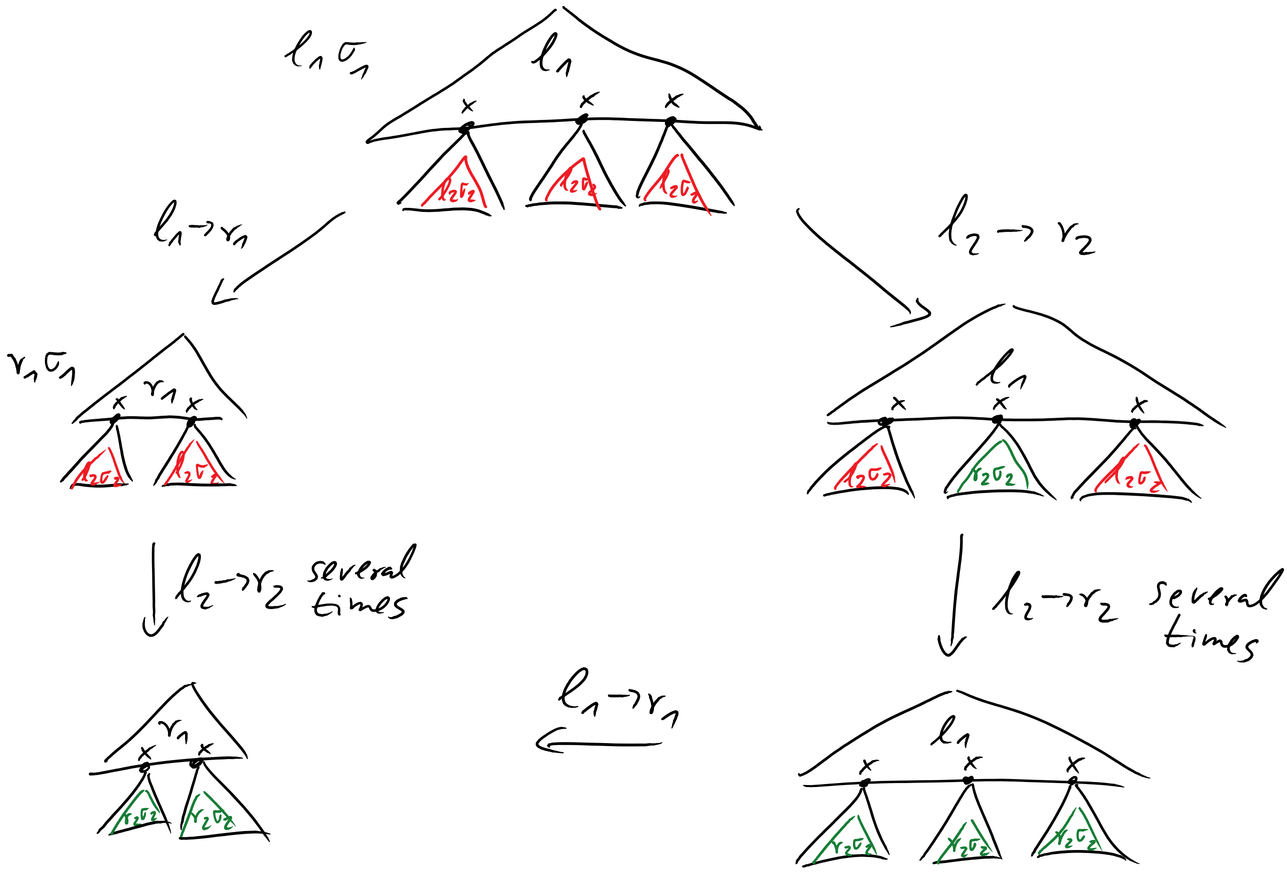
How "deep" is the subterm $l_2\sigma_2$ in $l_1\sigma_1$?

Case 2.1: $l_2\sigma_2$ is in "the substitution part" of $l_1\sigma_1$, i.e.: $\pi \notin Occ(l_1)$ or $l_1|_\pi \in \mathcal{V}$

So there is a $\pi' \in Occ(l_1)$

So there is a $\pi' \in Occ(l_1)$ where $l_1|_{\pi'} \in \mathcal{V}$ ← e.g., $l_1|_{\pi'} = x$ and $\pi = \pi' \pi''$.

The variable x (i.e., $l_1|_{\pi'}$) could occur several times in l_1. Then we have the following situation:



This situation is also harmless, i.e., this indeterminism can always be joined.

Ex:

$$f(f(y,e), i(f(y,e)))$$

(63) → e

(62) → $f(f(y,e), i(y))$

$e \downarrow *$

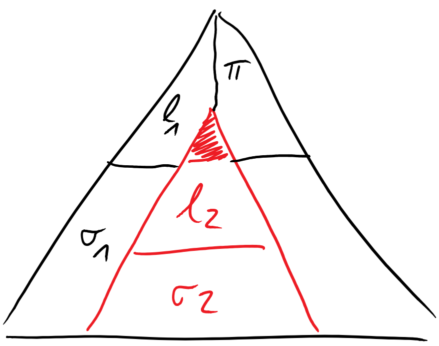
$f(f(y,e), i(y)) \downarrow$ (62)

$\downarrow \vartheta$ $\qquad\qquad\qquad\qquad$ $\downarrow (G2)$

$$e \xleftarrow{(G3)} f(y, i(y))$$

If two rules overlap on a variable position, then this is harmless for confluence: The resulting indeterminism can always be joined.

<u>Case 2.2</u> $\quad l_2\sigma_2$ is not in the "substitution part" of $l_1\sigma_1$, i.e., $\quad \pi_2 = \pi_1 \pi$ where $\pi \in Occ(l_1)$ with $l_1|_\pi \notin \mathcal{V}$.



This is called a <u>critical overlap</u>. We only have to regard these situations when checking for confluence.

$$l_1|_\pi \sigma_1 = l_2 \sigma_2$$

where $l_1|_\pi \notin \mathcal{V}$.

for rules $l_1 \to r_1$, $l_2 \to r_2$ where $l_1 \to r_1$ may also be the same rules as $l_2 \to r_2$.

If $l_1 \to r_1$ and $l_2 \to r_2$ have disjoint variables, then we can choose $\sigma_1 = \sigma_2$.

Critical Situation: 2 (variable-disj.) rules $l_1 \to r_1$, $l_2 \to r_2$

position π with $l_1|_\pi \notin \mathcal{V}$, $\underline{l_1|_\pi \sigma = l_2 \sigma}$. $\leftarrow$ $l_1|_\pi$ and l_2 unify

$$r_1\sigma \;\leftarrow\; \underline{l_1}\sigma = l_1[l_1|_\pi]_\pi \sigma = l_1[\underline{l_2}]_\pi \sigma \;\rightarrow\; l_1[r_2]_\pi \sigma$$

We have to check whether $r_1\sigma$ and $l_1[r_2]_\pi\sigma$ are joinable. Therefore, such pairs of terms are called <u>critical pairs</u>.

<u>Def 5.2.4</u> (Critical Pairs, Knuth + Bendix 1970)
Let $\mathcal{R}$ be a TRS with $l_1 \to r_1$, $l_2 \to r_2 \in \mathcal{R}$. Here, the variables are renamed such that $\mathcal{V}(l_1) \cap \mathcal{V}(l_2) = \emptyset$. Let $\pi \in Occ(l_1)$ with $l_1|_\pi \notin \mathcal{V}$ and let σ be a mgu of $l_1|_\pi$ and l_2. The rules $l_1 \to r_1$ and $l_2 \to r_2$ may be identical (up to variable renaming), but then we only regard overlaps at positions $\pi \neq \varepsilon$.
Then $\langle r_1\sigma, \; l_1[r_2]_\pi \sigma \rangle$ is called a critical pair of $\mathcal{R}$. $CP(\mathcal{R})$ is the set of all critical pairs of $\mathcal{R}$.

Every TRS has just finitely many critical pairs and $CP(\mathcal{R})$ can be computed automatically (since unification is decidable).

<u>Ex 5.2.5</u> Let $\mathcal{R} = \{(G1), (G2), (G3)\}$, i.e., $\mathcal{R}$ are the rules for the group example.

The subterm $f(y,z)$ of the lhs in $(G1)$ unifies with the lhs of $(G2)$; i.e., $mgu(f(y,z), f(x',e)) = \{x'/y, z/e\}$.

$$\begin{tikzcd} & f(x, \underline{f(y,e)}) \arrow[dl, "(G1)"'] \arrow[dr, "(G2)"] & \\ f(f(x,y),e) & & f(x,y) \end{tikzcd}$$

This results in the critical pair $\langle f(f(x,y),e), f(x,y) \rangle$

Moreover, $(G1)$ overlaps with $(G3)$:

$$\begin{tikzcd} & f(x, f(y, i(y))) \arrow[dl, "(G1)"'] \arrow[dr, "(G3)"] & \\ f(f(x,y),i(y)) & & f(x,e) \end{tikzcd}$$

Resulting critical pair:

$\langle f(f(x,y),i(y)), f(x,e) \rangle$

Finally, $(G1)$ overlaps with itself:

$$\underline{f(x, f(y,z))} \to f(f(x,y),z)$$
$$f(x', \underline{f(y',z')}) \to f(f(x',y'),z')$$

$f(y',z')$ unifies with $f(x, f(y,z))$

$$\begin{tikzcd} & f(x', \underline{f(x, f(y,z))}) \arrow[dl, "(G1)"'] \arrow[dr, "(G1)"] & \\ & & \end{tikzcd}$$

$$f(f(x',x), f(y,z)) \qquad\qquad f(x', f(f(x,y),z))$$

This results in the third crit. pair

$$\langle f(f(x',x), f(y,z)),\ f(x', f(f(x,y),z)) \rangle.$$

<u>Thm 5.2.6</u> (Critical Pair Lemma, Knuth+Bendix 1970)
A TRS $\mathcal{R}$ is locally confluent iff all its critical pairs are joinable.

<u>Proof</u>: follows from our previous considerations □

The critical pair lemma is suitable for automation:

- compute all critical pairs (automatable, because of UNIFY)
- for each critical pair $\langle s, t\rangle$ reduce both s and t to an arbitrary normal form $s \to^* s'$, $t \to^* t'$.
  (terminates if $\mathcal{R}$ is terminating).

If $s' = t'$, then $\langle s, t\rangle$ is joinable.

If $s' \neq t'$, then $\langle s,t\rangle$ might still be joinable, but we can stop and return "False" (the TRS is not confluent)

$$\begin{tikzcd}
 & p \arrow[dl] \arrow[dr] & \\
s \arrow[d, "*"'] & & t \arrow[d, "*"] \\
s' \arrow[rr, phantom, "\neq"] & & t'
\end{tikzcd}$$

Thus: there must be a non-joinable critical pair, but it could be a different one

Algorithm CONFLUENCE (see slide)

Thm 5.2.7 (Correctness of CONFLUENCE)

(a) The alg. CONFLUENCE always terminates and it is sound.

(b) Confluence is decidable for terminating TRSs.

Ex. 5.2.8 We check whether {(G1), (G2), (G3)} is confluent. The critical pairs resulting from the overlap of (G1) and (G3) and from (G1) and (G1) are joinable. But (G1) and (G3) lead to an overlap that results in two different normal forms $\Rightarrow$ TRS is not confluent.